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Research on Fractional Derivatives of Some Matrix Fractional Functions

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Abstract: **In this paper, based on Jumarie type of R-L fractional derivative, we find arbitrary order fractional derivative of two types of matrix fractional functions. Fractional Leibniz rule and a new multiplication of fractional analytic functions play important roles in this article. Moreover, our results are generalizations of ordinary calculus results.**

Keywords: **Jumarie type of R-L fractional derivative, matrix fractional functions, fractional Leibniz rule, new multiplication, fractional analytic functions.**

I. INTRODUCTION

Fractional calculus belongs to the field of mathematical analysis, involving the research and applications of arbitrary order integrals and derivatives. Fractional calculus originated from a problem put forward by L'Hospital and Leibniz in 1695. Therefore, the history of fractional calculus was formed more than 300 years ago, and fractional calculus and classical calculus have almost the same long history. Since then, fractional calculus has attracted the attention of many contemporary great mathematicians, such as N. H. Abel, M. Caputo, L. Euler, J. Fourier, A. K. Grunwald, J. Hadamard, G. H. Hardy, O. Heaviside, H. J. Holmgren, P. S. Laplace, G. W. Leibniz, A. V. Letnikov, J. Liouville, B. Riemann, M. Riesz, and H. Weyl. With the efforts of researchers, the theory of fractional calculus and its applications have developed rapidly. On the other hand, fractional calculus has wide applications in physics, mechanics, electrical engineering, viscoelasticity, biology, control theory, dynamics, economics, and other fields [1-16].

However, the definition of fractional derivative is not unique. Commonly used definitions include Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, Jumarie's modified R-L fractional derivative [17-21]. Because Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with classical calculus.

In this paper, based on Jumarie type of R-L fractional derivative and a new multiplication of fractional analytic functions, we obtain arbitrary order fractional derivative of the following two types of matrix fractional functions:

$$E_\alpha(tAx^\alpha)\otimes_\alpha cos_\alpha(tAx^\alpha),$$

$$E_\alpha(tAx^\alpha)\otimes_\alpha sin_\alpha(tAx^\alpha),$$

where $0<\alpha\leq 1$, t is a real number, and A is a matrix. Fractional Leibniz rule plays an important role in this article. In fact, our results are generalizations of classical calculus results.

II. PRELIMINARIES

Firstly, we introduce the fractional derivative used in this paper and its properties.

Definition 2.1 ([22])**:** Let $0<\alpha\leq 1$, and x_0 be a real number. The Jumarie type of Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^\alpha\right)[f(x)]=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^\alpha}dt\ , \tag{1}$$

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where $\Gamma(\ \ )$ is the gamma function. In addition, for any positive integer m, we define $\left({}_{x_0}D_x^{\alpha}\right)^m[f(x)] = \left({}_{x_0}D_x^{\alpha}\right)\left({}_{x_0}D_x^{\alpha}\right)\cdots\left({}_{x_0}D_x^{\alpha}\right)[f(x)]$, the m-th order α-fractional derivative of $f(x)$.

Proposition 2.2 ([23])**:** *If* α, β, x_0, C *are real numbers and* $\beta \geq \alpha > 0$, *then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{2}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[C] = 0. \tag{3}$$

Next, the definition of fractional analytic function is introduced.

Definition 2.3 ([24])**:** If x, x_0, and a_n are real numbers for all n, $x_0 \in (a,b)$, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as an α-fractional power series, i.e., $f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. Furthermore, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

In the following, we introduce a new multiplication of fractional analytic functions.

Definition 2.4 ([25])**:** Let $0 < \alpha \leq 1$, and x_0 be a real number. If $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}, \tag{4}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}. \tag{5}$$

Then we define

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} \otimes_\alpha \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} \\ &= \sum_{n=0}^{\infty} \frac{1}{\Gamma(n\alpha+1)}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)(x-x_0)^{n\alpha}. \end{aligned} \tag{6}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty} \frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n} \otimes_\alpha \sum_{n=0}^{\infty} \frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}. \end{aligned} \tag{7}$$

Definition 2.5 ([26])**:** If $0 < \alpha \leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} = \sum_{n=0}^{\infty} \frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}, \tag{8}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} = \sum_{n=0}^{\infty} \frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}. \tag{9}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha \circ g_\alpha)(x^\alpha) = f_\alpha(g_\alpha(x^\alpha)) = \sum_{n=0}^{\infty} \frac{a_n}{n!}\left(g_\alpha(x^\alpha)\right)^{\otimes_\alpha n}, \tag{10}$$

and

$$(g_\alpha \circ f_\alpha)(x^\alpha) = g_\alpha(f_\alpha(x^\alpha)) = \sum_{n=0}^{\infty} \frac{b_n}{n!}\left(f_\alpha(x^\alpha)\right)^{\otimes_\alpha n}. \tag{11}$$

Definition 2.6 ([27])**:** Let $0 < \alpha \leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ be two α-fractional analytic functions. Then $\left(f_\alpha(x^\alpha)\right)^{\otimes_\alpha n} = f_\alpha(x^\alpha)\otimes_\alpha \cdots \otimes_\alpha f_\alpha(x^\alpha)$ is called the n-th power of $f_\alpha(x^\alpha)$.

Definition 2.7 ([28])**:** If $0 < \alpha \leq 1$, x is a real variable and A is a matrix. The matrix α-fractional exponential function, matrix α-fractional cosine function, and matrix α-fractional sine function are defined as follows:

$$E_\alpha(Ax^\alpha) = \sum_{n=0}^{\infty} A^n \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(A\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha n}, \tag{12}$$

$$cos_\alpha(Ax^\alpha) = \sum_{n=0}^{\infty} A^{2n} \frac{(-1)^n x^{2n\alpha}}{\Gamma(2n\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(A\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2n}, \tag{13}$$

and

$$sin_\alpha(Ax^\alpha) = \sum_{n=0}^{\infty} A^{2n+1} \frac{(-1)^n x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(A\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (2n+1)}. \tag{14}$$

Theorem 2.8 (fractional Leibniz rule)**:** *If* $0 < \alpha \leq 1$, *and* $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ *are two* α*-fractional analytic functions at* $x = x_0$, *then*

$$\left({}_{x_0}D_x^\alpha\right)^m [f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha)] = \sum_{k=0}^{m} \binom{m}{k} \left({}_{x_0}D_x^\alpha\right)^{m-k} [f_\alpha(x^\alpha)] \otimes_\alpha \left({}_{x_0}D_x^\alpha\right)^k [g_\alpha(x^\alpha)], \tag{15}$$

where $\binom{m}{k} = \frac{m!}{k!(m-k)!}$.

Definition 2.9: The smallest positive real number T_α such that $E_\alpha(iT_\alpha) = 1$, is called the period of $E_\alpha(ix^\alpha)$.

III. MAIN RESULTS

In this section, we use fractional Leibniz rule to find arbitrary order fractional derivative of two types of matrix fractional functions.

Theorem 3.1: *If* $0 < \alpha \leq 1$, t *is a real number,* m *is a positive integer,* A, E *are matrices, and* E *is the unit matrix, then*

$$\left({}_0D_x^\alpha\right)^m [E_\alpha(tAx^\alpha) \otimes_\alpha cos_\alpha(tAx^\alpha)] = (tA)^m \sum_{k=0}^{m} \binom{m}{k} E_\alpha(tAx^\alpha) \otimes_\alpha cos_\alpha\left(tAx^\alpha + k \cdot \frac{T_\alpha}{4} E\right). \tag{16}$$

And

$$\left({}_0D_x^\alpha\right)^m [E_\alpha(tAx^\alpha) \otimes_\alpha sin_\alpha(tAx^\alpha)] = (tA)^m \sum_{k=0}^{m} \binom{m}{k} E_\alpha(tAx^\alpha) \otimes_\alpha sin_\alpha\left(tAx^\alpha + k \cdot \frac{T_\alpha}{4} E\right). \tag{17}$$

Proof Using fractional Leibniz rule yields

$$\begin{aligned}
&\left({}_0D_x^\alpha\right)^m [E_\alpha(tAx^\alpha) \otimes_\alpha cos_\alpha(tAx^\alpha)] \\
&= \sum_{k=0}^{m} \binom{m}{k} \left({}_{x_0}D_x^\alpha\right)^{m-k} [E_\alpha(tAx^\alpha)] \otimes_\alpha \left({}_{x_0}D_x^\alpha\right)^k [cos_\alpha(tAx^\alpha)] \\
&= \sum_{k=0}^{m} \binom{m}{k} (tA)^{m-k} E_\alpha(tAx^\alpha) \otimes_\alpha (tA)^k cos_\alpha\left(tAx^\alpha + k \cdot \frac{T_\alpha}{4} E\right) \\
&= (tA)^m \sum_{k=0}^{m} \binom{m}{k} E_\alpha(tAx^\alpha) \otimes_\alpha cos_\alpha\left(tAx^\alpha + k \cdot \frac{T_\alpha}{4} E\right).
\end{aligned}$$

And

$$\begin{aligned}
&\left({}_0D_x^\alpha\right)^m [E_\alpha(tAx^\alpha) \otimes_\alpha sin_\alpha(tAx^\alpha)] \\
&= \sum_{k=0}^{m} \binom{m}{k} \left({}_{x_0}D_x^\alpha\right)^{m-k} [E_\alpha(tAx^\alpha)] \otimes_\alpha \left({}_{x_0}D_x^\alpha\right)^k [sin_\alpha(tAx^\alpha)] \\
&= \sum_{k=0}^{m} \binom{m}{k} (tA)^{m-k} E_\alpha(tAx^\alpha) \otimes_\alpha (tA)^k sin_\alpha\left(tAx^\alpha + k \cdot \frac{T_\alpha}{4} E\right) \\
&= (tA)^m \sum_{k=0}^{m} \binom{m}{k} E_\alpha(tAx^\alpha) \otimes_\alpha sin_\alpha\left(tAx^\alpha + k \cdot \frac{T_\alpha}{4} E\right).
\end{aligned}$$

q.e.d.

IV. CONCLUSION

In this paper, based on Jumarie type of R-L fractional derivative and a new multiplication of fractional analytic functions, we use fractional Leibniz rule to obtain arbitrary order fractional derivative of two types of matrix fractional functions. Moreover, our results are generalizations of traditional calculus results. In the future, we will continue to use our methods to study the problems in engineering mathematics and fractional differential equations.

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